GENERALIZED GEOMETRY IN GRAVITY

RTG “MODELS OF GRAVITY” WORKSHOP

Eugenia Boffo

Jacobs University Bremen

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1. **Generalized Geometry**
   - I) approach: (extended) Riemannian geometry
   - II) approach: graded symplectic supermanifolds

2. \((g_{\mu \nu}, B_{\mu \nu}, \phi)\) **Fields in String Theory**

3. **Future Work**
   - Fermionic T-duality
   - \(\kappa\)-symmetry
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Leibniz algebroid: \((E, \rho, [\cdot, \cdot])\),
\(E \xrightarrow{\pi} M, \rho \in \text{Hom}(E, TM) \) “anchor”, \([\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\)

1. \([e, [e', e'']] = [[e, e'], e''] + [e', [e, e'']]\)
2. \([e, fe'] = (\rho(e)f)e' + f[e, e']\), \(e, e', e'' \in \Gamma(E)\)

\(\rho \in \text{Hom}(E, TM) \)

Lie algebroids: Leibniz algebroids with skew-symmetric brackets

\[
\exists \ d_E : \Omega^\bullet(E) \to \Omega^\bullet+1(E)
\]

Courant algebroid: \((E, \rho, [\cdot, \cdot], \langle \cdot, \cdot \rangle)\), Leibniz algebroid
\(\langle \cdot, \cdot \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M) \) “pairing” or \(g_E\),

1. \(\rho(e)\langle e', e'' \rangle = \langle [e, e'], e'' \rangle + \langle e', [e, e''] \rangle \implies \mathcal{L}_e g_E = 0\)
2. \(\langle [e, e], e' \rangle = \frac{1}{2} \rho(e')\langle e, e \rangle.\)
**BRACKETS**

\( \mathcal{E} \text{ exact Courant algebroid: } 0 \to T^*M \xrightarrow{j} E \xrightarrow{\rho} TM \to 0, \)
\( j := g_E^{-1} \circ \rho^T, \Im j = \ker \rho, \)

(Ševera) unique (up to an isomorphism) classification via \([H] \in H_3(M, \mathbb{R})\):

\[ \exists \text{ isotropic splitting } s : TM \to E, \langle s(X), s(Y) \rangle = 0 \text{ s.t.: } \]
\[ [s(X) + j(\xi), s(Y) + j(\eta)]_E = s([X, Y]_{\text{Lie}}) + j(\mathcal{L}_X \eta - i_Y d\xi - H(X, Y, \cdot)) \]
\[ \implies (E, \rho, [\cdot, \cdot]_E, \langle \cdot, \cdot \rangle_E) \cong (TM \oplus T^*M, \text{pr}_{TM}, [\cdot, \cdot]_D^H, \langle \cdot, \cdot \rangle_{TM \oplus T^*M}). \]

Twisting: if \( H \in \Omega^3_{\text{closed}}(M), B \in \Omega^2(M), \exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in O(d, d): \)

\[ \implies [\exp(B)e, \exp(B)e']_D^H = \exp(B) \left( [e, e']_D^H + dB \right) \]
\[ i_X i_Y H(Z) \equiv \langle [s(X), s(Y)], s(Z) \rangle \]

- **Dorfman bracket:**
  \( [V, W]_D := [X, Y]_{\text{Lie}} + \mathcal{L}_X \eta - i_Y d\xi. \)

- **Courant bracket:**
  \( [V, W]_C := \frac{1}{2} ([V, W]_E - [W, V]_E) \)

\( V, W \in TM \oplus T^*M \)
\( V = X + \xi \)
\( W = Y + \eta \)
I) approach: (extended) Riemannian geometry

Generalized metric $G_{\tau}$

for $\tau \in \text{End}(E)$, $\tau^2 = 1$ (involution),

$G_{\tau}(V, W) := \langle V, \tau(W) \rangle$

Theorem: if $E$ vector bundle with fiber-wise metric $\langle \cdot, \cdot \rangle_E$ of constant signature $(p, q)$, $\implies$ def. of $G_{\tau}$ is equivalent to def. of positive $C_+ \subseteq E$ of rank $p$

Theorem: if $L, L^*$ isotropic subbundles of $E$, $E = L \oplus L^*$, with $\langle \cdot, \cdot \rangle$ of signature $(n, n)$ $\implies$ $G_{\tau}$ is equivalent to unique $g \in \Gamma(S^2 L^*)$ and $B \in \Omega^2(L)$

$G_{\tau} = \begin{pmatrix} g -Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$, for $L = TM, L^* = T^*M$

$G_{\tau} \in O(d, d)/(O(d) \times O(d))$

for a more general reduction:

$O(t, s) \supset O(p, q) \times O(t-p, s-q)$

$\implies$

- $C_+ \cap T^*M = 0$
- $\text{rk} C_+ = \text{rk} E - \text{dim} M$
in extended tangent space:

- \( \nabla \) Courant connection if \( \nabla_e (fe') = f \nabla_e e' + \rho(e) (f) e' \) and
  \[ \langle \nabla_e e', e'' \rangle + \langle e', \nabla_e e'' \rangle = \rho(e) (\langle e', e'' \rangle) \]
- torsion \( T_\nabla \in \mathcal{T}_3 \) e.g. \( T_\nabla (e, e', e'') := \langle \nabla_e e' - \nabla_{e'} e - [e, e'], e'' \rangle + \langle \nabla_{e''} e, e' \rangle \)
- extended Riemann tensor, if \( R^{(0)}_\nabla (k', k, e, e') := \langle R(e, e') k, k' \rangle, \)
  \[ R_\nabla (k', k, e, e') := \frac{1}{2} \left\{ R^{(0)}_\nabla (k', k, e, e') + R^{(0)}_\nabla (e', e, k, k') + \langle \nabla_{e'} e, e' \rangle E \cdot \langle \nabla_{k'} k, k' \rangle \right\} \]

one amongst the possible generalizations of the std Levi-Civita connection (arXiv:1512.08522, Jurco, Vysoky):

\[
\nabla_0^X = \left( \nabla^L_C X + \frac{1}{6} g^{-1} H'(g^{-1}(\xi), \ast, \cdot) \right) \nabla^L_C X + \frac{1}{6} H'(X, g^{-1}(\ast), \cdot) - \frac{1}{3} g^{-1} H'(X, \xi^\lambda, \cdot) \nabla^L_C X + \frac{1}{6} H'(g^{-1}(\xi), g^{-1}(\ast), \cdot) \right) .
\]

\[
\text{Ric}_{\nabla_0} (e, e') := R_{\nabla_0} (e^\lambda, e, e_\lambda, e'), \quad R^G_{\nabla_0} := \text{Ric}_{\nabla_0} (G^{-1}(e^\lambda), e_\lambda),
\]
(II approach:) Recap of graded geometry

$(\mathbb{Z})$-graded vector space $A$: $A = \bigoplus_{i \in \mathbb{Z}} A_i$, over a field $k$ of characteristic zero
$\iff$ graded Poisson algebra of degree $n$, $\{\cdot, \cdot\}$ of degree $-n$, · zero-graded commutative product:

- $\{a, b\} = -(-1)^{(|a|+n)(|b|+n)} \{b, a\}$,
- $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)} \{a, \{b, c\}\}$
- $\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|b|(|a|+n)} b \cdot \{a, c\}$

Graded manifold $M$: its local coordinates have a degree $|\cdot|$
Graded vector bundle $E \xrightarrow{\pi} M$ : coords $\{x\}$ of deg 0 for $M$, coords $\{v\}$ of deg $n$ for $E$ $\implies$ its algebra of sections respects the grading

“coords even (in parity) have even degree, odd ones have odd degree” $\iff$ “N-manifold”
when $\exists Q \in \Gamma(TM)$, of degree 1, s.t. $[Q, Q] = 2Q^2 = 0$, $\implies$ “NQ-manifold”

to this setup, add a symplectic structure, i.e. a closed non-degenerate 2-form $\Omega$
of degree $n$, “Hamiltonian”
(II approach:) Poisson graded manifolds & Courant algebroids

**Theorem:** Symplectic $N$-manifolds of degree 2 are 1:1 with pseudo-Euclidean vector bundles [Roytenberg, arXiv:math/0203110v1]

$(M, \Omega)$ with $n = 2$ has homological vector field $Q$ given by $\Theta$ of degree 3,

$$\{\Theta, \Theta\} = 0,$$

$$Q := \{\Theta, \cdot\}$$

**Courant algebroids support grading**

**Theorem:** symplectic $NQ$-manifolds of degree 2 are 1:1 with Courant algebroids, in particular, $(T^*[2]T[1]M, pr_{TM}, [\cdot, \cdot], D, \langle \cdot, \cdot \rangle)$ is a Courant algebroid

Derived brackets:

$$\{\{\cdot, \Theta\}, \cdot\}$$

$$\{\{e, \Theta\}, f\} =: \rho(e) \cdot f$$

$$\{\{e, \Theta\}, e'\} =: [e, e']_{\text{Dorfman}}$$

$\{\cdot, \cdot\}$ is of degree -2, hence even
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$g_{\mu \nu}, B_{\mu \nu}, \phi$ FIELDS IN STRING THEORY (WITH APPROACH II)

base manifold $M$, $T^*[2] T[1] M$ total space, $\{x, \xi, \theta, p\}$

$C^\infty(M) \ni f = f(x)$,

$\Gamma(T[1] M \oplus T^*[1] M) \ni V, W = Y^i \chi_i + \eta^i \theta^i \ (\rightarrow \xi_\alpha := (\chi_i, \theta^i), W = W^\alpha \xi_\alpha, \tilde{\xi}^\beta$ dual)

Poisson graded structure given by:

canonical Poisson brackets

\[
\begin{align*}
\{g, f\} &= 0 \\
\{p_i, f\} &= \partial_i f \\
\{p_i, \xi_\alpha\} &= 0 \\
\{\xi_\alpha, \xi_\beta\} &= \eta_{\alpha \beta} \\
\{p_i, p_j\} &= 0
\end{align*}
\]

deformation

\[
\begin{align*}
\{g, f\} &= 0 \\
\{p_i, f\} &= \partial_i f \\
\{p_i, \xi_\alpha\} &= \nabla_i \xi_\alpha \\
\{\xi_\alpha, \xi_\beta\} &= G_{\alpha \beta}(x) \\
\{p_i, p_j\} &= R_{ij} = 0
\end{align*}
\]

\[G = \begin{pmatrix} 2g(x) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g(x) - B(x) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

$G$ corresponds to $e^{(g+B)} \chi_k = \chi_k + (g_{kl} + B_{kl}) \theta^l$

it is possible to canonically associate a cubic hamiltonian, $\Theta$, through the homological vector field as $Q = \{\Theta, -\}$,

\[\Theta = \tilde{\xi}^\alpha \rho(\xi_\alpha) + \frac{1}{3!} C^{\alpha \beta \gamma}(x) \xi_\alpha \xi_\beta \xi_\gamma,
\]

for $\rho : E \to TM$ anchor, $C^{\alpha \beta \gamma}$ 3-tensor ("fluxes").
(INTERMEDIATE) RESULTS

given that $\nabla_{\rho(\xi_\alpha)} \xi_\beta = W_{\alpha\beta}^{\gamma} \xi_\gamma = \{\rho(\xi_\alpha), \xi_\beta\}$, $W = \partial (g + B)$

$$
\{\{\{V, \Theta\}, W\}, U\} = \langle \nabla_V W, U \rangle - \langle \nabla_W V, U \rangle + \langle \nabla_U V, W \rangle + C(V, W, U)
=:\langle [V, W]', U \rangle
$$

$$
[V, W]' = [\rho(V), \rho(W)]_{\text{Lie}} + T(V, W) + \langle \nabla_V W, W \rangle + \text{“fluxes”}
=:\langle [V, W]', U \rangle
$$

$\langle [V, W]', U \rangle$ is Koszul formula for $g + B$:

$$
2g(\tilde{\nabla}_X Y, Z) = g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) + \partial_X (g(Y, Z) + B(Y, Z)) + \partial_Y (g(X, Z) + B(X, Z)) - \partial_Z (g(X, Y) + B(X, Y))
\Longleftrightarrow
W_{\alpha\beta}^{\gamma} - W_{\beta\alpha}^{\gamma} + W_{\gamma\alpha\beta} = 2\Gamma_{\beta(\gamma\alpha)} + H_{\alpha\beta\gamma}
$$

gravity sector of string theory for dilaton $\phi = 0$ reconstructed from Ricci tensor $R_{\mu\nu}$ for generalized $T^*[1]M \oplus T[1]M$, contracted with $(g^{\mu\nu} + B^{\mu\nu})$:

$$
\mathcal{I}_{\sigma-st.thy} = \frac{1}{4\pi\kappa^2} \int \text{Vol}(g) \left( R - \frac{1}{12} H^2 \right)
$$
(INTERMEDIATE) RESULTS

To obtain the $\sigma$-model for the full closed string sector (26-dim for bosonic string, 10-dim for RNS-string), seek for deformations whose connection is given by

$$\Gamma_{\nu\delta}^{\text{L.C.}} \beta + \frac{1}{2} H_{\nu\delta} \beta - \frac{2}{d-1} \partial_{\delta} \phi \delta_{\nu} \beta,$$

reconstruction of $\mathcal{L}_{\sigma-\text{st.thy}}$ goes as before

$$\mathcal{L} = e^{-2\phi} \left( R - \frac{1}{12} H^2 + 4(\partial \phi)^2 \right)$$

where the beta functions are:

$$\beta_{\mu\nu}^g + \beta_{\mu\nu}^B = R_{\mu\nu}^{\text{L.C.}} - \frac{1}{4} H_{\mu\lambda\epsilon} H_{\nu}^{\lambda\epsilon} + 2 \nabla_{\mu} \partial_{\nu} \phi - \frac{1}{2} \nabla^{\lambda} H_{\lambda\mu\nu} + \partial^{\lambda} \phi H_{\lambda\mu\nu}$$
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3. Fermionic T-duality

We argue that transformations (fermionic T-dualities) of string fluxes could be related to a deformation of a Courant algebroid and its bracket.

\[ \mathcal{A}^{GS} = \int d^2 z B_{00} (Y) \partial \theta^0 \bar{\partial} \theta^0 + L_{0M} (Y) \partial \theta^0 \bar{\partial} Y^M + L_{M0} (Y) \partial Y^M \bar{\partial} \theta^0 + L_{MN} (Y) \partial Y^M \bar{\partial} Y^N, \]

isometry \( \theta^0 \to \theta^0 + c \iff \) conservation in one direction (\( \exists \) a killing spinor). Integrating it out (by means of its e.o.m.'s) one ends with the same type of theory IIA (or IIB) if the \( F^i \) R-R fields and the dilaton \( \phi \) change according to

\[
\exp (\phi') F' = \exp (\phi) F \pm 32 \sum_{I,J=1}^{N} \left( \varepsilon_I \otimes \hat{\varepsilon}_J \right) M_{IJ} \\
\phi' = \phi + \frac{1}{2} \text{Tr} (\log M)^{-1}
\]

\( M^{-1} \) given by:

\[
\partial_a \left( \begin{array}{c} M^{-1} \\ \end{array} \right)_{IJ} = i \varepsilon_I \gamma_a \varepsilon_J + i \hat{\varepsilon}_I \gamma_a \hat{\varepsilon}_J,
\]

and \( \left( \begin{array}{c} \varepsilon \\ \hat{\varepsilon} \end{array} \right) \)

SO(2)-invariant Majorana-Weyl spinor (same chirality)
4. $\kappa$-SYMMETRY

We argue that $\kappa$-symmetry of RNS string theory could have a geometrical meaning in the generalized geometry setup.

Spacetime supersymmetric Nambu-Goto string action:

$$\mathcal{S}_{NG1} = -\frac{\alpha'}{\pi} \int d^2 \sigma \sqrt{-\text{det} \left[ \left( \partial_\alpha X^\mu - \bar{\theta}^A \Gamma_\mu \partial_\alpha \theta^A \right) \left( \partial_\beta X^\mu - \bar{\theta}^A \Gamma_\mu \partial_\beta \theta^A \right) \right]} ,$$

global supersymmetry: $\delta \theta^A = \epsilon^A a$, $\delta X^\mu = \bar{\epsilon}^a \Gamma_\mu \theta^A$, with algebra $[\delta_1, \delta_2] \theta = 0$, $[\delta_1, \delta_2] X^\mu = -2 \bar{\epsilon}_1 \Gamma_\mu \epsilon_2$.

Equations of motion: $\dot{P} = 0$, $\dot{\theta} = 0 \implies$ only half of the dofs of $\theta$ are genuine.

$$\implies \mathcal{S}_{WZ2} = -m \int d\tau \bar{\theta} \Gamma_{11} \partial_\tau \theta$$

is needed.

This preserves a hidden local symmetry called $\kappa$-symmetry.
**SUMMARY**

- **generalized geometry:** Courant algebroids for a “doubled” total space $TM \oplus T^* M$, usually extended Riemannian geometry is assigned;

- 1:1 correspondence with graded symplectic manifolds of deg 2, endowed with $Q$ homological vector field, and $\Theta$ hamiltonian

  - aim: a full derivation of closed strings in low-energy string theory $\leftarrow$ so far, we have a connection, still a good deformation of the Poisson bracket, s.t. $\phi$ is included, is missed;

  - aim: a study of the transformation of fluxes in supersymmetric string theory: fermionic T-duality & $\kappa$-symmetry
Thanks for your attention