Generalized geometry in gravity

RTG "Models of Gravity" workshop

Eugenia Boffo

Jacobs University Bremen

February 21, 2018

Table of Contents

TABLE OF CONTENTS

ALGEBROIDS

Leibniz algebroid: $(E, \rho, [\cdot,\cdot])$,
$E \xrightarrow{\pi} M$, $\rho \in \mathrm{Hom}(E, TM)$ *"anchor"*, $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

1. $[e, [e', e'']] = [[e, e'], e''] + [e', [e, e'']]$
2. $[e, \mathrm{f}e'] = (\rho(e)\mathrm{f})e' + \mathrm{f}[e, e']$,

$e, e', e'' \in \Gamma(E)$
$\mathrm{f} \in C^\infty(M)$

Lie algebroids: Leibniz algebroids with skew-symmetric brackets

$$\longrightarrow \quad \exists \ \mathrm{d}_E : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$$

Courant algebroid: $(E, \rho, [\cdot,\cdot], \langle\cdot,\cdot\rangle)$, Leibniz algebroid
$\langle\cdot,\cdot\rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ "pairing" or g_E,

1. $\rho(e)\langle e', e''\rangle = \langle [e, e'], e''\rangle + \langle e', [e, e'']\rangle \implies \mathscr{L}_e g_E = 0$
2. $\langle [e, e], e'\rangle = \frac{1}{2}\rho(e')\langle e, e\rangle$.

BRACKETS

E *exact* Courant algebroid: $0 \to T^*M \xrightarrow{j} E \xrightarrow{\rho} TM \to 0$,
$j := g_E^{-1} \circ \rho^T$, $\mathrm{Im} j = \ker \rho$,

(Ševera) unique (up to an isomorphism) classification via $[H] \in H_3(M, \mathbb{R})$:

$\exists$ isotropic splitting $s : TM \to E$, $\langle s(X), s(Y) \rangle = 0$ s.t.:
$[s(X) + j(\xi), s(Y) + j(\eta)]_E = s([X,Y]_{\mathrm{Lie}}) + j(\mathscr{L}_X \eta - i_Y d\xi - H(X,Y,\cdot))$

$$\Longrightarrow (E, \rho, [\cdot,\cdot]_E, \langle \cdot,\cdot \rangle_E) \cong (TM \oplus T^*M, pr_{TM}, [\cdot,\cdot]_D^H, \langle \cdot,\cdot \rangle_{TM \oplus T^*M}).$$

Twisting: if $H \in \Omega^3_{closed}(M)$, $B \in \Omega^2(M)$, $\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in \mathrm{O}(d,d)$:

$$\Longrightarrow [\exp(B)e, \exp(B)e']_D^H = \exp(B)\left([e,e']_D^{H+dB}\right)$$

$$i_X i_Y H(Z) \equiv \langle [s(X), s(Y)], s(Z) \rangle$$

- Dorfman bracket:
  $[\mathrm{V},\mathrm{W}]_D := [X,Y]_{\mathrm{Lie}} + \mathscr{L}_X \eta - i_Y d\xi.$
- Courant bracket: $[\mathrm{V},\mathrm{W}]_C := \frac{1}{2}([\mathrm{V},\mathrm{W}]_E - [\mathrm{W},\mathrm{V}]_E)$

$\mathrm{V}, \mathrm{W} \in TM \oplus T^*M$
$\mathrm{V} = X + \xi$
$\mathrm{W} = Y + \eta$

(I APPROACH:) GENERALIZED METRIC $\mathbf{G}_\tau$

$$\text{for } \tau \in \mathrm{End}(E),\ \tau^2 = 1 \text{ (involution)},$$
$$\mathbf{G}_\tau(\mathrm{V},\mathrm{W}) := \langle \mathrm{V}, \tau(\mathrm{W})\rangle$$

Theorem: *if E vector bundle with fiber-wise metric $\langle\cdot,\cdot\rangle_E$ of constant signature (p,q), $\Longrightarrow$ def. of $\mathbf{G}_\tau$ is equivalent to def. of positive $C_+ \subseteq E$ of rank p*

Theorem: *if L, L^* isotropic subbundles of E, $E = L \oplus L^*$, with $\langle\cdot,\cdot\rangle$ of signature $(n,n) \Longrightarrow \mathbf{G}_\tau$ is equivalent to unique $g \in \Gamma(S^2 L^*)$ and $B \in \Omega^2(L)$*

$$\mathbf{G}_\tau = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \quad \text{for } L = TM,\ L^* = T^*M$$

$$\mathbf{G}_\tau \in \mathrm{O}(d,d)/(\mathrm{O}(d)\times \mathrm{O}(d))$$

for a more general reduction:
$O(t,s) \supset O(p,q) \times O(t-p,s-q)$ $\Longrightarrow$

- $C_+ \cap T^*M = 0$
- $\mathrm{rk}C_+ = \mathrm{rk}E - \dim M$

(I approach:) Bismut connection $\widetilde{\nabla}^0$

in extended tangent space:

- ∇ Courant connection if $\nabla_e(\mathrm{f}e') = \mathrm{f}\nabla_e e' + \rho(e)(\mathrm{f})e'$ and $\langle \nabla_e e', e'' \rangle + \langle e', \nabla_e e'' \rangle = \rho(e)(\langle e', e'' \rangle)$
- torsion $T_\nabla \in \mathscr{T}_3$ e.g. $T_\nabla(e,e',e'') := \langle \nabla_e e' - \nabla_{e'} e - [e,e'], e'' \rangle + \langle \nabla_{e''} e, e' \rangle$
- extended Riemann tensor, if $R^{(0)}_\nabla(k',k,e,e') := \langle R(e,e')k, k' \rangle$, $R_\nabla(k',k,e,e') := \frac{1}{2}\left\{ R^{(0)}_\nabla(k',k,e,e') + R^{(0)}_\nabla(e',e,k,k') + \langle \nabla_{e_\lambda} e, e' \rangle_E \cdot \langle \nabla_{k_\lambda} k, k' \rangle \right\}$

one amongst the possible generalizations of the std Levi-Civita connection (arXiv:1512.08522, Jurco, Vysoky):

$$\widetilde{\nabla}^0_X = \begin{pmatrix} \nabla^{LC}_X + \frac{1}{6} g^{-1} H'(g^{-1}(\xi), \star, \cdot) & -\frac{1}{3} g^{-1} H'(X, g^{-1}(\star), \cdot) \\ -\frac{1}{3} H'(X, \star, \cdot) & \nabla^{LC}_X + \frac{1}{6} H'(g^{-1}(\xi), g^{-1}(\star), \cdot) \end{pmatrix}.$$

$$\mathrm{Ric}_{\widetilde{\nabla}^0}(e,e') := R_{\widetilde{\nabla}^0}(e^\lambda, e, e_\lambda, e'), \quad \mathrm{R}^{\mathbf{G}}_{\widetilde{\nabla}^0} := \mathrm{Ric}_{\widetilde{\nabla}^0}(\mathbf{G}^{-1}(e^\lambda), e_\lambda),$$

(II APPROACH:) RECAP OF GRADED GEOMETRY

($\mathbb{Z}$-)graded vector space A: $A = \oplus_{i\in\mathbb{Z}} A_i$, over a field k of characteristic zero $\Longleftrightarrow$ graded Poisson algebra of degree n, $\{\cdot,\cdot\}$ of degree $-n$, $\cdot$ zero-graded commutative product:

- $\{a,b\} = -(-1)^{(|a|+n)(|b|+n)}\{b,a\}$,
- $\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+n)(|b|+n)}\{a,\{b,c\}\}$
- $\{a,b\cdot c\} = \{a,b\}\cdot c + (-1)^{|b|(|a|+n)} b\cdot\{a,c\}$

$|\cdot| :=$ degree

Graded manifold M: its local coordinates have a degree $|\cdot|$

Graded vector bundle $E \xrightarrow{\pi} M$: coords $\{x\}$ of deg 0 for M, coords $\{v\}$ of deg n for $E \Longrightarrow$ its algebra of sections respects the grading

"coords even (in parity) have even degree, odd ones have odd degree" $\Longleftrightarrow$ *"N-manifold"*

when $\exists\, Q \in \Gamma(TM)$, of degree 1, s.t. $[Q,Q] = 2Q^2 = 0$, $\Longrightarrow$ *"NQ-manifold"*

to this setup, add a sympletic structure, i.e. a closed non-degenerate 2-form Ω of degree n, "Hamiltonian"

(II APPROACH:) POISSON GRADED MANIFOLDS & COURANT ALGEBROIDS

Theorem: *Symplectic N-manifolds of degree 2 are 1:1 with pseudo-Euclidean vector bundles* [Roytenberg, arXiv:math/0203110v1]

(M,Ω) with $n=2$ has homological vector field Q given by Θ of degree 3, $\{\Theta,\Theta\}=0$,

$$Q := \{\Theta,\cdot\}$$

COURANT ALGEBROIDS SUPPORT GRADING

Theorem: symplectic NQ-manifolds of degree 2 are $1:1$ with Courant algebroids,
in particular, $(T^*[2]T[1]M, pr_{TM}, [\cdot,\cdot]_D, \langle\cdot,\cdot\rangle)$ is a Courant algebroid

Derived brackets : $\{\{\cdot,\Theta\},\cdot\}$

$$\{\{e,\Theta\},f\} =: \rho(e)\cdot f$$
$$\{\{e,\Theta\},e'\} =: [e,e']_{\text{Dorfman}}$$

$\{\cdot,\cdot\}$ is of degree -2, hence even

TABLE OF CONTENTS

$(g_{\mu\nu}, B_{\mu\nu}, \phi)$ FIELDS IN STRING THEORY (WITH APPROACH II)

base manifold M, $T^*[2]T[1]M$ total space, $\{x, \xi, \theta, p\}$

$C^\infty(M) \ni \mathrm{f} = \mathrm{f}(x)$,

$\Gamma(T^*[2]M) \ni \upsilon, \omega = \omega^i p_i$ $\quad$ $\Gamma(T[1]M \oplus T^*[1]M) \ni V, W = Y^i \chi_i + \eta_i \theta^i$ ($\rightarrow \xi_\alpha := (\chi_i, \theta^i), W = W^\alpha \xi_\alpha$, $\tilde{\xi}^\beta$ dual)

Poisson graded structure given by:

canonical Poisson brackets $\quad$ deformation

$$\begin{cases} \{\mathrm{g}, \mathrm{f}\} = 0 \\ \{p_i, \mathrm{f}\} = \partial_i \mathrm{f} \\ \{p_i, \xi_\alpha\} = 0 \\ \{\xi_\alpha, \xi_\beta\} = \eta_{\alpha\beta} \\ \{p_i, p_j\} = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \{\mathrm{g}, \mathrm{f}\} = 0 \\ \{p_i, \mathrm{f}\} = \partial_i \mathrm{f} \\ \{p_i, \xi_\alpha\} = \nabla_i \xi_\alpha \\ \{\xi_\alpha, \xi_\beta\} = G_{\alpha\beta}(x) \\ \{p_i, p_j\} = R_{ij} = 0 \end{cases}$$

$$G = \begin{pmatrix} 2g(x) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g(x) - B(x) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g(x) + B(x) & 1 \end{pmatrix}$$

G corresponds to $e^{(g+B)} \chi_k = \chi_k + (g_{kl} + B_{kl}) \theta^l$

it is possible to canonically associate a cubic hamiltonian, Θ, through the homological vector field as $Q = \{\Theta, -\}$,

$$\Theta = \tilde{\xi}^\alpha \rho(\xi_\alpha) + \frac{1}{3!} C^{\alpha\beta\gamma}(x) \xi_\alpha \xi_\beta \xi_\gamma,$$

for $\rho : E \rightarrow TM$ anchor, $C^{\alpha\beta\gamma}$ 3-tensor (*"fluxes"*).

(INTERMEDIATE) RESULTS

given that $\nabla_{\rho(\xi_\alpha)}\xi_\beta = \mathbb{W}_{\alpha\beta}{}^{\gamma}\xi_\gamma = \{\rho(\xi_\alpha), \xi_\beta\}$, $\mathbb{W} = \partial(g+B)$

$$\begin{aligned}\{\{\{V,\Theta\},W\},U\} &= \langle \nabla_V W, U\rangle - \langle \nabla_W V, U\rangle + \langle \nabla_U V, W\rangle + C(V,W,U)\\ &=: \langle [V,W]', U\rangle\end{aligned}$$

$$\begin{aligned}[V,W]' &= [\rho(V),\rho(W)]_{\text{Lie}} + T(V,W) + \langle \nabla . V, W\rangle + \text{“fluxes”}\\ &=: [\rho(V),\rho(W)]_{\text{Lie}} + \langle \widetilde{\nabla} V, W\rangle + \text{“fluxes”}\end{aligned}$$

$\langle [V,W]', U\rangle$ is Koszul formula for $g+B$:
$2g(\widetilde{\nabla}_X Y, Z) = g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) + \partial_X(g(Y,Z)+B(Y,Z)) + \partial_Y(g(X,Z)+B(X,Z)) - \partial_Z(g(X,Y)+B(X,Y))$

$\Longleftrightarrow$

$\mathbb{W}_{\alpha\beta\gamma} - \mathbb{W}_{\beta\alpha\gamma} + \mathbb{W}_{\gamma\alpha\beta} = 2\Gamma_{\beta(\gamma\alpha)} + H_{\alpha\beta\gamma}$

$\downarrow$

gravity sector of string theory for dilaton $\phi = 0$ reconstructed from Ricci tensor $R_{\mu\nu}$ for generalized $T^*[1]M \oplus T[1]M$, contracted with $(g^{\mu\nu} + B^{\mu\nu})$:

$$\mathscr{S}_{\sigma-\text{st.thy}} = \frac{1}{4\pi\kappa^2}\int \text{Vol}(g)\left(R - \frac{1}{12}H^2\right)$$

(INTERMEDIATE) RESULTS

To obtain the σ-model for the full closed string sector (26-dim for bosonic string, 10-dim for RNS-string), seek for deformations whose connection is given by

$$\Gamma^{\text{L.C.}\,\beta}_{\nu\delta} + \frac{1}{2} H_{\nu\delta}{}^{\beta} - \frac{2}{d-1} \partial_\delta \phi \, \delta_\nu{}^\beta ,$$

reconstruction of $\mathscr{L}_{\sigma-\text{st.thy}}$ goes as before

$$\Longrightarrow (g^{\mu\nu} + B^{\mu\nu}) e^{-2\phi} \left(\beta^g_{\mu\nu} + \beta^B_{\mu\nu} - \frac{4}{d-1} \partial_\mu \phi \partial_\nu \phi \right)$$

$$\mathscr{L} = e^{-2\phi} \left(R - \frac{1}{12} H^2 + 4(\partial\phi)^2 \right)$$

where the beta functions are:
$\beta^g_{\mu\nu} + \beta^B_{\mu\nu} = R^{\text{L.C}}_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\varepsilon} H_\nu{}^{\lambda\varepsilon} + 2\nabla_\mu \partial_\nu \phi - \frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + \partial^\lambda \phi H_{\lambda\mu\nu}$

TABLE OF CONTENTS

3. Fermionic T-duality

We argue that transformations (fermionic T-dualities) of string fluxes could be related to a deformation of a Courant algebroid and its bracket.

$$\mathscr{S}_{\sigma}^{GS} = \int d^2z\, B_{00}(Y)\,\partial\theta^0\bar{\partial}\theta^0 + L_{0M}(Y)\,\partial\theta^0\bar{\partial}Y^M + L_{M0}(Y)\,\partial Y^M\bar{\partial}\theta^0 + L_{MN}(Y)\,\partial Y^M\bar{\partial}Y^N\,,$$

isometry $\theta^0 \to \theta^0 + c \Longrightarrow$ conservation in one direction ($\exists$ a killing spinor). Integrating it out (by means of its e.o.m.'s) one ends with the same type of theory IIA (or IIB) if the F^i R-R fields and the dilaton ϕ change according to

$$\exp(\phi')F' = \exp(\phi)F \pm 32 \sum_{I,J=1}^{N} (\varepsilon_I \otimes \hat{\varepsilon}_J)\, M_{IJ}$$

$$\phi' = \phi + \frac{1}{2}\mathrm{Tr}\,(\log M)^{-1}$$

M^{-1} given by:

$$\partial_a\left(M^{-1}\right)_{IJ} = i\varepsilon_I\gamma_a\varepsilon_J \mp i\hat{\varepsilon}_I\gamma_a\hat{\varepsilon}_J\,,$$

and $\begin{pmatrix}\varepsilon\\ \hat{\varepsilon}\end{pmatrix}$

SO(2)-invariant Majorana-Weyl spinor (same chirality)

4. κ-SYMMETRY

We argue that κ-symmetry of RNS string theory could have a geometrical meaning in the generalized geometry setup.

Spacetime supersymmetric Nambu-Goto string action:

$$\mathscr{S}_{NG\,1} = -\frac{\alpha'}{\pi}\int d^2\sigma\,\sqrt{-\det\left[\left(\partial_\alpha X^\mu - \bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A\right)\left(\partial_\beta X_\mu - \bar{\theta}^A\Gamma_\mu\partial_\beta\theta^A\right)\right]}\,,$$

global supersymmetry: $\delta\theta^{Aa} = \varepsilon^{Aa}$, $\delta X^\mu = \bar{\varepsilon}^a\Gamma^\mu\theta^A$, with algebra $[\delta_1,\delta_2]\,\theta = 0$, $[\delta_1,\delta_2]\,X^\mu = -2\bar{\varepsilon}_1\Gamma^\mu\varepsilon_2$.

Equations of motion: $\dot{P} = 0$, $\dot{\theta} = 0 \implies$ only half of the dofs of θ are genuine.

$$\implies \mathscr{S}_{WZ\,2} = -m\int d\tau\,\bar{\theta}\,\Gamma_{11}\partial_\tau\theta$$

is needed.

This preserves a hidden local symmetry called κ-symmetry.

SUMMARY

- *generalized geometry: Courant algebroids for a "doubled" total space* $TM \oplus T^*M$, usually extended Riemannian geometry is assigned;
- *1:1 correspondance with graded symplectic manifolds of deg 2, endowed with Q homological vector field, and Θ hamiltonian*
  - aim: a full derivation of closed strings in low-energy string theory $\leftarrow$ so far, we have a connection, still a good deformation of the Poisson bracket, s.t. ϕ is included, is missed;
  - aim: a study of the transformation of fluxes in supersymmetric string theory: fermionic T-duality & κ-symmetry

Thanks for your attention