The zeroth law of black hole thermodynamics revisited

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Zeroth law of thermodynamics

... defines a notion of equilibrium:

... and of temperature.
(intensive quantity!)
Zeroth law of thermodynamics

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Black hole thermodynamics I

Analogy to ordinary TD:

- energy... $U \leftrightarrow M$ ... mass
- temperature... $T \leftrightarrow \kappa$ ... surface gravity
- entropy... $S \leftrightarrow A$ ... horizon area

First law of BH TD:

$$dM = TdS + \Omega dJ + \Phi dQ$$

Smarr relation: [1]

$$(D - 3)M = (D - 2)TS + (D - 2)\Omega J + (D - 3)\Phi Q$$
Conjugate variables: intensive ↔ extensive!

\[ T = \frac{\partial M}{\partial S}, \quad \Omega = \frac{\partial M}{\partial J}, \quad \Phi = \frac{\partial M}{\partial Q} \]

Fundamental relation:

\[ S(M, J, Q) = \pi \left( 2M^2 - Q^2 + 2\sqrt{+M^4 - M^2Q^2 - J^2} \right) \]

\[ \implies \text{entropy representation:} \]

\[ \left\{ S, M, J, Q, \frac{1}{T}, -\frac{\Omega}{T}, -\frac{\Phi}{T} \right\} \]
Black hole thermodynamics: AdS extension

AdS black holes: \[1, 2\]

- pressure... \( P \leftrightarrow \Lambda \) ... cosmological constant
- volume... \( V \leftrightarrow V_{th} \) ... thermal volume of the bh

First law: derived from geometric properties.

\[
dM = TdS + \Omega dJ + \Phi dQ + VdP \equiv dH
\]

\( \Rightarrow \) mass equals enthalpy: \( M = H = U + PV \)
Phase transitions in black hole systems?

- **Hawking-Page phase transition**: \([3, 4]\)
  between thermal AdS space and large black hole solution. Calculating the free energy!

- **Kerr-AdS**: \([1, 2]\)
  Van der Waals-type phase transition in P-V-diagram!
  Swallow tail in Gibbs free energy.
Questions:

Is the P-V-diagram at constant T transferable to black hole thermodynamics?

⇓

Is T a valid intensive quantity to measure thermodynamic equilibrium?

Suppose:

$S$... thermodyn. potential, $\{q^i\}$... extensive vars., $\{p_i\}$... intensive vars.
Quasi-homogeneity: [5, 6] of degree \( r \) and type \( \beta \)

\[
S(\lambda_1^{\beta_1}q^1, \ldots, \lambda_n^{\beta_n}q^n) = \lambda^r S(q^1, \ldots, q^n)
\]

Homogeneity of degree 1:

\[
S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)
\]

Intensive thermodynamic variables:

\[
p_i(q^i) \equiv \left. \frac{\partial S(q^i)}{\partial q^i} \right|_{q_j}
\]

\[
\Rightarrow \quad p_i(\lambda_1^{\beta_1}q^j) = \lambda^{r - \beta_i} p_i(q^j)
\]

... quasi-homogeneous of degree \( r - \beta_i \).
Euler’s theorem

For quasi-homogeneous functions holds:

\[ rS(q^{j}) = \sum_{i=1}^{n} \beta_{i} [q^{i}p_{i}(q^{j})] \]

**Smarr** relation for black holes \( \equiv \) **Euler** relation

\[ S = \frac{1}{2} \frac{1}{T} M - \frac{1}{2} \frac{\Omega}{T} J - \frac{\Phi}{T} Q + \frac{1}{2} \frac{P}{T} V \]

\[ \uparrow \]

read off degree and type!
... obtained by combining Smarr relation with first law. Usually:

\[
U d \left( \frac{1}{T} \right) - V d \left( \frac{P}{T} \right) + N d \left( \frac{\mu}{T} \right) = 0
\]

**Generalized Gibbs-Duhem:**

\[
\sum_{i=1}^{n} \left[ (\beta_i - r)p_i (q^i) dq^i + \beta_i q^i dp_i (q^i) \right] = 0
\]

\[\Rightarrow (1)\] Variation of extensive quantities \(q^i\)
\[\Rightarrow (2)\] Usual variation of intensive quantities \(p_i\)
Mathematical inconsistency

Is the generalized Gibbs-Duhem relation fulfilled in the quasi-homogeneous case?

Definition of equilibrium:

\[ dp_i = 0 \implies (2) = 0 \]

- Homogeneous TD: \( \beta_i = r = 1 \implies (1) = 0 \)
- Quasi-homogeneous TD:

\[ \beta_i, r \neq 1 \implies (1) \neq 0 \]

(1) related to latent heat in phase transitions!
Redefine equilibrium!

\[ \tilde{p}_i(q^j) \equiv \left[ (q^i)^{\beta_i - r} \right]^{1/\beta_i} p_i(q^j) \]

... obtained by solving the generalized Gibbs-Duhem relation.
... one particular solution, more general intensities possible.
... quasi-homogeneous of degree 0.
... reduce to \( p_i \) in the case \( \beta_i = r = 1 \).
Rewrite the generalized Gibbs-Duhem relation:

\[
\sum_{i=1}^{n} \left[ (\beta_i - r) p_i(q^j) dq^i + \beta_i q^i dp_i(q^j) \right] = \equiv \sum_{i=1}^{n} \beta_i \left( q_i \right)^{r/\beta_i} d\tilde{p}_i(q^j) = 0
\]

Fulfilled by new definition of equilibrium!
Schwarzschild entropy: \( S(M) = 4\pi M^2 \Rightarrow r = 2, \beta = 1 \)

Usual temperature:

\[
\frac{1}{T} = \frac{\partial S}{\partial M} = 8\pi M
\]

... homogeneous of degree 1! Not truly intensive!

**New definition of equilibrium:**

\[
\frac{1}{\tilde{T}} = \left(M^{\beta-r}\right)^{1/\beta} \frac{1}{T} = \frac{1}{TM} = 8\pi
\]

... trivially intensive!
The popular case: Kerr-AdS

Fundamental potential: $H(S, P, J = 1)$

- "Intensive" quantities $T, P$ not (quasi-)homogeneous of degree 0.
- Explicit calculation: generalized Gibbs-Duhem not fulfilled
- Phase transition: swallow tail in Gibbs free energy
  $$G(T, P, J) = U - TS + PV$$
The popular case: Kerr-AdS

Introduce new **intensive** quantities:

\[ \tilde{T} = TU \quad \text{and} \quad \tilde{P} = PUV^{1/3} \]

... quasi-homogeneous of degree 0!

No PT!
Another interesting case: Hawking-Page?

PT in AdS black holes: $J = 0$

But: $S$ and $V$ are not independent variables!

$$
\left( \frac{3V}{4\pi} \right)^2 = \left( \frac{S}{\pi} \right)^3
$$

$\Rightarrow P(V, T)$ and $G(T, P)$ cannot be calculated!

PT structure has been found in different way!

Method not applicable?
Introduction of true intensities is important in quasi-homogeneous thermodynamics.

- Revise notion of TD equilibrium in general?
- Thermodynamic response functions?
- Extend methodology?
- Fundamental meaning?

Thanks!

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