The zeroth law of black hole thermodynamics revisited

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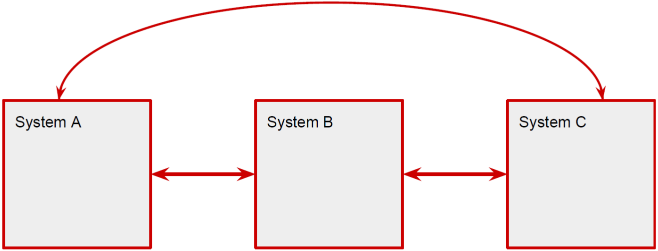
Zeroth law of thermodynamics

... defines a notion of equilibrium:



... and of temperature.
(intensive quantity!)

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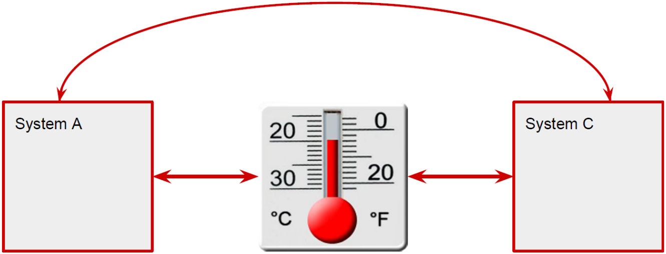

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Black hole thermodynamics I

Analogy to ordinary TD:

$$\begin{aligned} \text{energy... } U &\leftrightarrow M \text{ ... mass} \\ \text{temperature... } T &\leftrightarrow \kappa \text{ ... surface gravity} \\ \text{entropy... } S &\leftrightarrow A \text{ ... horizon area} \\ \ldots\ldots &\leftrightarrow \ldots\ldots \end{aligned}$$

First law of BH TD:

$$dM = TdS + \Omega dJ + \Phi dQ$$

Smarr relation: [1]

$$(D-3)M = (D-2)TS + (D-2)\Omega J + (D-3)\Phi Q$$

Black hole thermodynamics II

Conjugate variables: intensive $\leftrightarrow$ extensive!

$$T = \frac{\partial M}{\partial S}\,, \quad \Omega = \frac{\partial M}{\partial J}\,, \quad \Phi = \frac{\partial M}{\partial Q}$$

Fundamental relation:

$$S(M,J,Q) = \pi\left(2M^2 - Q^2 + 2\sqrt{+M^4 - M^2Q^2 - J^2}\right)$$

$\Longrightarrow$ entropy representation:

$$\left\{S, M, J, Q, \frac{1}{T}, -\frac{\Omega}{T}, -\frac{\Phi}{T}\right\}$$

Black hole thermodynamics: AdS extension

AdS black holes: [1, 2]

pressure... $P \quad \leftrightarrow \quad \Lambda$ ... cosmological constant

volume... $V \quad \leftrightarrow \quad V_{th}$ ... thermal volume of the bh

First law: derived from geometric properties.

$$dM = TdS + \Omega dJ + \Phi dQ + VdP \equiv dH$$

$\implies$ mass equals enthalpy: $M = H = U + PV$

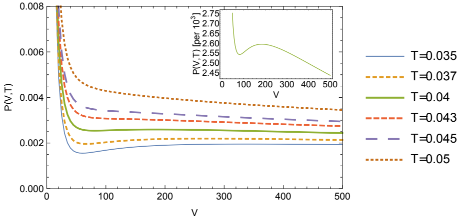
Phase transitions in black hole systems?

- ## Hawking-Page phase transition: [3, 4]

  between thermal AdS space and large black hole solution. Calculating the free energy!

- ## Kerr-AdS: [1, 2]

  Van der Waals-type phase transition in P-V-diagram! Swallow tail in Gibbs free energy.



Questions:

Is the P-V-diagram at constant T transferable to black hole thermodynamics?

$$\Updownarrow$$

Is T a valid intensive quantity to measure thermodynamic equilibrium?

Suppose:

S... thermodyn. potential, $\{q^i\}$... extensive vars., $\{p_i\}$... intensive vars.

(Quasi-)Homogeneity in thermodynamics

Quasi-homogeneity: [5, 6] **of degree r and type β**

$$S(\lambda^{\beta_1} q^1, ..., \lambda^{\beta_n} q^n) = \lambda^r S(q^1, ..., q^n)$$

Homogeneity of degree 1: $S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$

Intensive thermodynamic variables:

$$p_i\left(q^j\right) \equiv \frac{\partial S\left(q^j\right)}{\partial q^i} \quad \Longrightarrow \quad p_i\left(\lambda^{\beta_j} q^j\right) = \lambda^{r-\beta_i} p_i(q^j)$$

... quasi-homogeneous of degree $r - \beta_i$.

Euler's theorem

For quasi-homogeneous functions holds:

$$rS(q^j) = \sum_{i=1}^{n} \beta_i \left[q^i p_i(q^j) \right]$$

Smarr relation for black holes $\equiv$ Euler relation

$$S = \frac{1}{2}\frac{1}{T}M - \frac{1}{2}\frac{\Omega}{T}J - \frac{\Phi}{T}Q + \frac{1}{2}\frac{P}{T}V$$

$$\Downarrow$$

read off degree and type!

Gibbs-Duhem relation

... obtained by combining Smarr relation with first law.
Usually:

$$U d\left(\frac{1}{T}\right) - V d\left(\frac{P}{T}\right) + N d\left(\frac{\mu}{T}\right) = 0$$

Generalized Gibbs-Duhem:

$$\sum_{i=1}^{n} \Big[\underbrace{(\beta_i - r) p_i(q^j) dq^i}_{(1)} + \underbrace{\beta_i q^i dp_i(q^j)}_{(2)} \Big] = 0$$

$\Longrightarrow$ (1) Variation of extensive quantities q^i
$\Longrightarrow$ (2) Usual variation of intensive quantities p_i

Mathematical inconsistency

Is the generalized Gibbs-Duhem relation fulfilled in the quasi-homogeneous case?

Definition of equilibrium:

$$dp_i = 0 \Longrightarrow (2) = 0$$

- Homogeneous TD: $\beta_i = r = 1 \Longrightarrow (1) = 0$
- Quasi-homogeneous TD:

$$\beta_i, r \neq 1 \Longrightarrow (1) \neq 0$$

(1) related to latent heat in phase transitions!

Generalized intensities

Redefine equilibrium!

$$\tilde{p}_i(q^j) \equiv \left[\left(q^i \right)^{\beta_i - r} \right]^{1/\beta_i} p_i(q^j)$$

... obtained by solving the generalized Gibbs-Duhem relation.

... one particular solution, more general intensities possible.

... quasi-homogeneous of degree 0.

... reduce to p_i in the case $\beta_i = r = 1$.

Generalized Gibbs-Duhem relation

Rewrite the generalized Gibbs-Duhem relation:

$$\sum_{i=1}^{n}\left[\,(\beta_i - r)\,p_i(q^j)dq^i + \beta_i q^i dp_i(q^j)\right] =$$

$$\equiv \sum_{i=1}^{n} \beta_i \left(q^i\right)^{r/\beta_i} \underbrace{d\tilde{p}_i(q^j)}_{=0} = 0$$

Fulfilled by new definition of equilibrium!

The simplest case: not quite significant?

Schwarzschild entropy: $S(M) = 4\pi M^2 \Rightarrow r = 2,\ \beta = 1$

Usual temperature:

$$\frac{1}{T} = \frac{\partial S}{\partial M} = 8\pi M$$

... homogeneous of degree 1! Not truly intensive!

New definition of equilibrium:

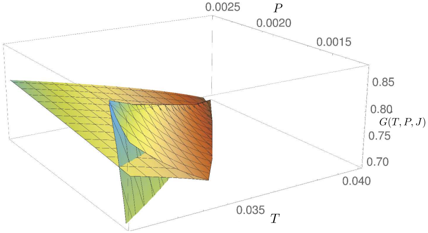
$$\frac{1}{\tilde{T}} = \left(M^{\beta - r}\right)^{1/\beta} \frac{1}{T} = \frac{1}{TM} = 8\pi$$

... trivially intensive!

The popular case: Kerr-AdS

Fundamental potential: $H(S, P, J = 1)$

- "Intensive" quantities T, P not (quasi-)homogeneous of degree 0.
- Explicit calculation: generalized Gibbs-Duhem not fulfilled
- Phase transition: swallow tail in Gibbs free energy $G(T, P, J) = U - TS + PV$



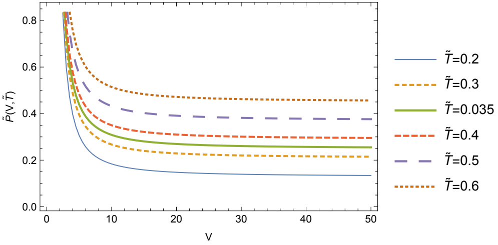
The popular case: Kerr-AdS

Introduce new intensive quantities:

$$\tilde{T} = TU \quad \text{and} \quad \tilde{P} = PUV^{1/3}$$

... quasi-homogeneous of degree 0!

No PT!



Another interesting case: Hawking-Page?

PT in AdS black holes: $J = 0$

But: S and V are not independent variables!

$$\left(\frac{3V}{4\pi}\right)^2 = \left(\frac{S}{\pi}\right)^3$$

$\Longrightarrow P(V,T)$ and $G(T,P)$ cannot be calculated!

PT structure has been found in different way!
Method not applicable?

Conclusions

$\Longrightarrow$ Introduction of true intensities is important in quasi-homogeneous thermodynamics.

- Revise notion of TD equilibrium in general?
- Thermodynamic response functions?
- Extend methodology?
- Fundamental meaning?

Thanks!

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